Restricted Non-interactive Zero Knowledge Proofs

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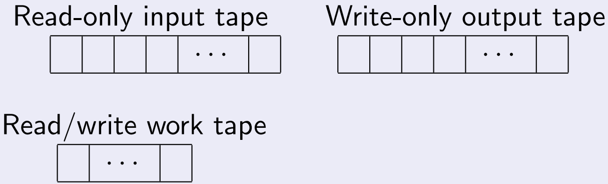
Outline

Log-space computation

Definition

Figure 1: Log-space transducer



Has only logarithmic work space, but still relatively powerful

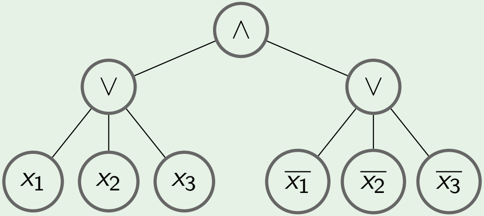
- Closed under function composition
- Closed under complement
- Can run for (at most) polynomial time

L is the class of problems computable by some log-space transducer

Circuit complexity

Example

Figure 2: Output node is 1 iff input has at least one 0 and one 1



Circuit (family) classes:

- NC^0: Fan-in 2, constant depth, poly-size
- AC^0: Unbounded fan-in, constant depth, poly-size
- Projection: no gates, only wires from input to output (possibly negated)

(NISZK) Non-interactive, statistical zero knowledge proof

Intuition: prover wants to prove input x has some property to the verifier without revealing additional information

Definition

An NISZK proof system consists of four main parts:

- Prover: very powerful machine, outputs some distribution of proofs given (x, σ).
- Verifier: randomized, limited machine which almost always accepts if correct proof is provided for (x, σ) (completeness), and almost always rejects on a fake or insufficient proof (soundness)
- Simulator: used to guarantee zero knowledge. Should output a distribution of proofs statistically similar to the prover's distribution
- Reference string σ: uniformly random string provided to verifier and prover to use during the proof - "shared randomness"

Known results

$\mathsf{NISZK_L}$

NISZK, but with log-space verifier and simulator. Introduced in 2020 paper by Allender and REU students [All+21].

Was shown to have two complete problems - SDU_{NC^0} and EA_{NC^0} - which are modified versions of complete problems for NISZK.

Definition

Promise-$\mathsf{EA}_{\mathsf{NC}^0}$: a promise problem over pairs (C, k), where C is an NC^0_4 circuit and k an integer.

$$(C, k) \in \mathsf{EA}_{YES} \text{ if the Shannon entropy of } C \text{ is } \geq k + 1$$
$$(C, k) \in \mathsf{EA}_{NO} \text{ if the Shannon entropy of } C \text{ is } \leq k - 1.$$

Not much else was known about this class outside of this paper.

Perfect randomized encodings

Introduced in [AIK06] to show that NC^0 had OWFs iff more powerful classes like NL did.

Definition

Adapted from [AII+21, Definition 27] (perfect randomized encoding):

Let $f : \{0,1\}^n \to \{0,1\}^{l(n)}$ be a function. We say that $\hat{f} : \{0,1\}^n \times \{0,1\}^m \to \{0,1\}^s$ is a perfect uniform randomized encoding of f with blowup b if it is:

- **Input Independent:** for every $x, x' \in \{0,1\}^n$ such that $f(x) = f(x')$, $\hat{f}(x, U_m)$ and $\hat{f}(x', U_m)$ are identically distributed.
- **Output Disjoint:** for every $x, x' \in \{0,1\}^n$ such that $f(x) \neq f(x')$, $\operatorname{supp} \hat{f}(x, U_m) \cap \operatorname{supp} \hat{f}(x', U_m) = \emptyset$.
- **Uniform:** for every $x \in \{0,1\}^n$ the random variable $\hat{f}(x, U_m)$ is uniform over $\operatorname{supp} \hat{f}(x, U_m)$.
- **Balanced:** for every $x, x' \in \{0,1\}^n$ $|\operatorname{supp} \hat{f}(x, U_m)| = |\operatorname{supp} \hat{f}(x', U_m)| = b$.

New results

$\mathsf{NISZK_L} = \mathsf{NISZK_{AC_0}}$

Since $\mathsf{AC_0} \in \mathsf{L}$, it is clear that $\mathsf{NISZK_{AC_0}} \in \mathsf{NISZK_L}$.
To prove: $\mathsf{NISZK_L} \in \mathsf{NISZK_{AC_0}}$ The Entropy Approximation Problem in $\mathsf{NC_0}$, denoted $EA_{\mathsf{NC_0}}$, is a complete problem for $\mathsf{NISZK_L}$.
We will show that $EA_{\mathsf{NC_0}} \in \mathsf{NISZK_{AC_0}}$.

Proving $EA \in$ NISZK

Transform an instance (C, k) of EA of length s into a distribution Z by

- Taking poly(s) copies of X
- hashing
- repeat items 1 and 2

The Problem: AC_0 circuits cannot compute hash functions. They can be computed in logspace.

Solution

Theorem

There exists a constant c such that, for every $k(n) > n/\operatorname{poly}(\log n)$, and every $r(n) \in [\Omega(\log n), k(n)/c]$, extraction of $(1+c)\cdot r(n)$ bits that are $\epsilon(n) = \frac{1}{n^3}$ close to $\{0,1\}^{(1+c)\cdot r(n)}$ in total variation distance is possible in uniform AC_0 using a seed of length $r(n)$.

Theorem

There is a polynomial time computable function that takes an instance $(X, m-a)$ of EA_{NC_0} and a parameter s, and produces a distribution Z on $\{0,1\}^l$ such that:

1. *If $H(X) > m-a+1$, then Z has statistical difference at most $1/\operatorname{poly}(s)$ from the uniform distribution on $\{0,1\}^l$.*
2. *If $H(X) < m-a-1$, then the support of Z is at most a 2^{-s} fraction of $\{0,1\}^l$.*

Non Interactive Proof System

1. Let Z be the distribution on $\{0,1\}^l$ obtained from $(X, m-a)$ taking s to be the total description length of $(X, m-a)$ in bits. Let $\sigma_1, \sigma_2, \ldots, \sigma_{s/\log s} \in \{0,1\}^l$ be the reference strings. The verifier sends $\sigma_1, \sigma_2, \ldots, \sigma_{s/\log s}$ to the prover.
2. The prover picks an i at random from $i \in \{0, 1, \ldots, s/\log(s)\}$ such that $|\{r_i | Z(r_i) = \sigma_i\}| \neq \phi$. Then, after fixing i,it picks a random r_i from $\{r_i | Z(r_i) = \sigma_i\}$. It sends r_i to the verifier.
3. V accepts if $Z(r_i) = \sigma_i$.

Simulator

1. Let Z be obtained from $(X, m-a)$ as in the proof system.
2. Sample an i uniformly at random from $\{1, 2, \ldots, s/\log s\}$.
3. For this index i,sample r_i at random, and compute $Z(r_i) = \sigma_i$.
4. For all $j \in \{1, 2, \ldots, i-1, i+1, \ldots, s/\log(s)\}$, sample σ_j uniformly at random.
5. Output $(r_i, \sigma_1, \ldots \sigma_i = Z(r_i), \ldots \sigma_{s/\log(s)})$

Completeness

Claim

If $H(X) > m - a + 1$, then the verifier accepts with probability $\geq 1 - \frac{1}{2^s}$.

Proof.

If $H(X) > m - a + 1$, then $TV(Z, U_{\{0,1\}^l}) \leq \frac{1}{\text{poly}(s)}$. Thus, for a given i

$$
\begin{aligned}
P(\exists r_i | Z(r_i) = \sigma_i) &\geq 1 - P(\nexists r_i | Z(r_i) = \sigma_i) \\
&\geq 1 - \prod_{i=1}^{s/\log(S)} \frac{1}{\text{poly}(s)} \\
&\geq 1 - \frac{1}{\text{poly}(s)^{s/\log s}} \\
&\geq 1 - \frac{1}{2^s}
\end{aligned}
$$

□

Soundness

Claim

If $H(X) < m - a - 1$, then the verifier accepts with probability $\leq \frac{1}{2^s}$.

Proof.

If $H(X) < m - a - 1$, then, by Lemma 0.2, the support of Z is at most a 2^{-s} fraction of $\{0,1\}^l$. Thus,

$$\begin{aligned} P(\textit{verifier accepts}) &= P(\exists i | Z(r_i) = \sigma_i) \\ &\leq \sum_{i=1}^{s/\log s} \frac{1}{2^s} \\ &\leq \frac{s}{\log s} \cdot \frac{1}{2^s} \\ &\approx \frac{1}{2^s} \end{aligned}$$

□

Construction of Distribution Z by AC_0 Circuits

We let $s \approx m$ be the length of the description of an instance of (X, k) of EA.

Let the threshold for the EA problem be $k = m - a$, where a is a small constant $\in (0, 1)$.

STEP 1: Many copies of distribution X

Let m (resp. n) be the number input (resp. output) gates to X. We take $q = 4sm^2$ independent copies of X to create distribution X'. Observe that $H(X') = q \cdot H(X)$. For every $x, P(X = x) \geq \frac{1}{2^m}$. So the flattening lemma implies that X' is $\delta = \sqrt{q} \cdot m = 2\sqrt{s} \cdot m^2$ flat.
Thus,

1. if $H(X) > k + 1$, then $H(X') > q \cdot k + q > qk$.
2. If $H(X) < k - 1$, then $H(X') < q \cdot k - q$.

STEP 2: Using AC_0 Randomness Extractor on X'

Randomness source: $r \in \{0,1\}^{qk/c}$, where c is the constant mentioned in theorem 1. Consider distribution Y on $E(x', r) : \{0,1\}^{qm} \times \{0,1\}^{qk/c} \to \{0,1\}^{qk+qk/c}$.

1. If $H(X) > k + 1$, then the statistical difference of Y from the uniform distribution over $\{0,1\}^{qk+qk/c}$ is at most $1/(qm)^3$.
2. If $H(X) < qk - 1$, then $H(Y) < q \cdot k - q + qk/c$.

STEP 3: Many copies of distribution Y

1. If $H(X) > k+1$, then Y' has statistical difference at most $q' \cdot \frac{1}{(qm)^3} = (4s(qm)^2) \cdot \frac{1}{(qm)^3} = \frac{4s}{qm} = \frac{1}{m^3} = \mathcal{O}\frac{1}{\text{poly}(s)}$.
2. If $H(X) < k-1$, then $H(Y') < q' \cdot H(Y) < q' \cdot q \cdot k \cdot (\frac{c+1}{c}) - q' \cdot q$.

STEP 4: Using AC_0 randomness extractor on Y'.

Let Z be the resulting distribution.
$Z(r') = (Y'(r'), E(r', r))$
If $H(X) > k + 1$,the statistical distance of Z from uniform is $\approx 1/\operatorname{poly}(s)$.
If $H(X) < k - 1$,
$S1 : \{(Y'(r'), E(r', r)) | P(Y'(r') = y') \leq 2^{-N-2M}\}$
$S2 : \{(Y'(r'), E(r', r)) | 2^{-N-2M} \leq P(Y'(r') = y') \leq 2^{-N}\}$
$S3 : \{(Y'(r'), E(r', r)) | P(Y'(r') = y') \geq 2^{-N}\}$
For $i \in \{1, 2, 3\}, |S_i|/|D| = \frac{1}{2^s}$, where D is the uniform distribution.
Thus, $Z = S1 \cup S2 \cup S3$
$|Z|/|D| = \frac{1}{2^s}$

$\mathsf{NISZK}_\mathsf{L} = \mathsf{NISZK}_\mathsf{NL}$

For an arbitary problem $\Pi \in \mathsf{NISZK}_\mathsf{NL}$, let (S, P, V) be its protocol. Then, we will run the following:

Algorithm 1: $M_x(s)$

Data: $x \in \Pi \cup \bar{\Pi}, s$ coin flips, (S, V) the simulator and verifier
$(\sigma, p) \leftarrow S(x)$ `// under coin flip` s
if $V(\sigma, p) = 1$ **then**
  return σ;
else
  return $0^{|\sigma|}$;

(Convince yourself that $M_x(s)$ is NL-computable). Now, on $x \in \Pi$, w.h.p. $M_x(s)$ is uniform over its support, and on $x \notin \Pi$ w.h.p. $M_x(s)$ has small support. We want to find an encoding of $M_x(s)$ in NC^0 with similar entropy.

$\mathsf{NISZK_L} = \mathsf{NISZK_{NL}}$

Idea: [RA97] shows that for any instance of

$$\mathsf{PATH} := \{\langle G, s, t\rangle : s \rightsquigarrow t\}$$

there is a random weight assignment $W = \{w_i : i \in [n^2]\}, w_i : E(G) \to [4n^2]$ such that with probability exponentially close to 1: $\exists i$ minimal such that (G, w_i) has a unique minimum path from s to t.
This reduces any NL problem to a UL problem, where UL is a class already known to have perfect randomized encodings in NC^0. This completes the high level idea of our proof.
We also proved that $\mathsf{NISZK_L} = \mathsf{NISZK_{PM}}$ using a similar proof, but due to time constraints it won't be explained here.

Summary

Presented results:

- $\mathsf{NISZK}_{\mathsf{AC}^0} = \mathsf{NISZK}_{\mathsf{L}} = \mathsf{NISZK}_{\mathsf{NL}}$

Going forward:

- $\mathsf{NISZK}_{\mathsf{L}} \stackrel{?}{=} \mathsf{NISZK}_{\mathsf{DET}}$
- $\mathsf{OWFs} \in \mathsf{NC}^0 \stackrel{?}{\leftrightarrow} \mathsf{OWFs} \in DET$

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